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STABILITY ANALYSIS AND GUARANTEED DOMAIN OF ATTRACTION FOR A CLASS OF HYBRID SYSTEMS: AN LMI APPROACH [1]

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LCMI/DAS REPORT

December 2000

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Stability Analysis and Guaranteed Domain of Attraction for a class of Hybrid Systems: an LMI Approach

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Abstract

This paper presents sufficient conditions for the robust stability problem and the determination of an estimate of the domain of attraction for switched piecewise affine systems, a special class of Hybrid Systems. This class of systems are represented by an affine differential equation of the type $\dot{x} = A(\delta)x + b(\delta)$, where x denotes the continous state vector and δ is a vector of logical variables that modifies the local model of the system in accordance with the continuous dynamics. Using a Lyapunov function of the type $v(x) = x'\mathcal{P}(x)x$, we present LMI conditions that, when feasible, guarantee local properties of the origin of the switched system. A switched system and a linear system with control saturation are used to illustrate the results.

1 Introduction

Piecewise affine systems, sometimes called piecewise linear systems, has been investigated during the past years due to wide applicability in a large class of nonlinear systems. For example: linear closed-loop systems with bounded inputs can be modeled as piecewise affine systems [1]; switched controllers, such as the gain-scheduling [2] and variable structure control [3], are based on piecewise ideas; nonlinear systems approximated by local models, such as the fuzzy [4] and ARMAX [5] modeling techniques are described by piecewise affine systems. Moreover, some classes of hybrid systems, a mathematical model of dynamical process with continuous and discrete variables, can be viewed as switched systems [6]. Generalizing, from the results of [7] where the Lyapunov's indirect method was extended to nonlinear switched systems, piecewise affine systems may be used to model a large class of switched systems.

The wide range of applications of piecewise affine systems leaded to the appearence of different approaches. For instance, saturated systems can be analysed via circle and Popov criteria [8] or by piecewise approach [1]. Focusing on the piecewise Lyapunov function techniques, there are different classes of Lyapunov functions, such as the linear [9, 10] and quadratic [11, 12, 13] functions. In the last years, many results on the stability of piecewise affine systems using piecewise Lyapunov functions have appeared [14, 15, 16]. However, the main problem with Lyapunov theory is that there are no general method for searching Lyapunov functions. Besides, the guarateed domain of attraction problem must be considered due to the nonlinear behaviour of the piecewise affine systems for no global equilibrium points [8].

Since the development of *interior-point methods* for solving semidefinite programming problems and the publication of the book [17], the LMI framework has been largely used to solve many

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control problems. The basic idea is to reformulate the control problems in terms of linear matrix inequalities (LMIs), i.e. positive semidefinite constraints involving symmetric matrices which are affine in the decision variables. Recently, new advances in the LMI framework are shown in [18], such as analysis and synthesis of parameter-varying systems, optimization of integral quadratic constraints, impulse differential systems, parameter dependent Lyapunov functions, $\mathcal{H}_2$ filter/control problem, gain-scheduled and multiobjective control.
At the end of this decade, many construtive methods based on LMI techniques were proposed for testing the stability and computing an estimate of the domain of attraction of switched system using piecewise quadratic Lyapunov functions, i.e., the state space is divided in several regions and each region is associated with a different system model and a suitable quadratic Lyapunov function [19, 1].
Differently from the above approaches, we propose in this paper a single Lyapunov function that has the following features. First, instead of quadratic functions we use a polynomial function of the form $v(x) = x'\mathcal{P}(x)x$, where $\mathcal{P}(x)$ is itself a quadratic matrix function of the state x. Second, the Lyapunov function candidate is the same for all the state space partitions. Via LMI tools this work addresses the problems of stability analysis and determining a guaranteed domain of attraction for a class of Hydrid Systems which are described by an affine differential equation where the linear local models are modified by state-dependent logical variables.
This paper is organized as follows. The next section states the problem of concern and introduces the class of hybrid/switched systems. In section 3, the stability results are summarized by the theorem 1, which propose LMI conditions for the stability problem. The problem of finding a suitable estimate of the domain of attraction is addressed in the section 4. In the sequel, two numerical examples are presented and some concluding remarks end the paper.
The notation used here is usual: $\mathbb{R}^n$ denotes the set of n-dimensional real vectors, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, $0_{n\times m}$ is the $n \times m$ matrix of zeros and 0_n is the $n \times n$ matrix of zeros). For a real matrix S, the notation $S > 0$ means that S is symmetric and positive definite. The matrix and vector dimensions are omitted whenever they can be determined from the context.

2 Problem Statement

Consider the following class of systems

$$\dot{x}(t) = A_i x(t) + b_i \ , \quad x \in X_i \ , \quad i \in \{1, 2, \dots, n\} \ , \quad t > 0 \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the continuous component of the state taking values in the subsets $X_i \subset \mathbb{R}^{n_x}$, i denotes the discrete variable which belongs to the subset of integers $\{i = 1, \dots, n\}$, $A_i \in \mathbb{R}^{n_x \times n_x}$, for $i = 1, \dots, n$, are given constant matrices and $b_i \in \mathbb{R}^{n_x}$, for $i = 1, \dots, n$, are given constant vectors.
The time argument t is often omitted in the sequel for readability.
It is well-known that systems with switching vector fields require special attention in their mathematical description [20]. To avoid undesirable behaviour in our system model, we will assume with respect to system (1) that:

A 1 *$\mathcal{B}_x$ is a given polytope representing a neighbourhood of the origin of $\mathbb{R}^{n_x}$ in which the local stability property will be analyzed and the origin, $x = 0$, is an equilibrium point of system (1).*

A 2 *The subsets X_i have non-empty disjoint interiors and satisfy $X_1 \cup \ldots \cup X_n \equiv \mathcal{B}_x \subseteq \mathbb{R}^{n_x}$, these imply that there are not overlappings between the regions of our model.*

A 3 *For all* $x(0) = x_0 \in \mathcal{B}_x$ *, the trajectory* $x(t)$ *that belongs to* $\mathcal{B}_x$ *is continuous.*

A 4 *The changes in the discrete variable* i *are governed only by the continuous state component, in other words, the system (1) can be characterized as an autonomous switched system.*

A 5 *There is a finite number of transitions in any finite interval of time.*

A 6 *On the boundaries between adjacent regions* X_k*,* $k = i, \ldots, i + \bar{k}$*, there are no sliding behaviour, i.e. if* $x(t^-) \in X_i \cap X_j$ *then* $x(t) \notin X_i \cap X_j$*, where* $x(t^-)$ *denotes the trajectory position before the time* t*.*

For convenience, the discrete variable i will be associated to a set of logical variables δ_i given by the vector $\delta = \left[\begin{array}{cccc} \delta_1 & \delta_2 & \ldots & \delta_n \end{array} \right]' \in \mathbb{R}^n$ such that $\delta_i \in \{0, 1\}$, $i \in \{1, \ldots, n\}$, and if $x(t) \in X_i$ then $\delta_i = 1$ and $\delta_j = 0$ for all $j \neq i$, $j \in \{1, \ldots, n\}$.
When the trajectory $x(t)$ passes from the set X_i to X_j at an instant t_s, the logical variables δ_i and δ_j are definied as $\delta_i = 1$ and $\delta_j = 0$, i.e. if $x(t = t_s) \in X_i \cap X_j$ and $x(t^-) \in X_i$ then $\delta_i = 1$ and $\delta_j = 0$.
In this way, observe that $\sum_{i=1}^{n} \delta_i = 1$ and thus $\delta_n = 1 - \sum_{i=1}^{n-1} \delta_i$.
With the above notation we may recast (1) as

$$\dot{x} = \left(\sum_{i=1}^{n} \delta_i A_i \right) x + \sum_{i=1}^{n} \delta_i b_i \ , \ x \in \mathbb{R}^{n_x} \ , \ \delta \in \tilde{\mathcal{B}}_\delta = \left\{ \delta : \left\{ \begin{array}{l} \delta_i = 1 \text{ if } \left\{ \begin{array}{l} x(t) \in X_i \text{ ; or} \\ x(t) \in X_i \cap X_j \text{ and} \\ x(t^-) \in X_i \ . \end{array} \right. \\ \delta_i = 0 \quad \text{otherwise.} \end{array} \right. \right\} \tag{2}$$

Let $\mathcal{B}_\delta$ be the following polytope

$$\mathcal{B}_\delta = \mathbf{Co}\{\tilde{\mathcal{B}}_\delta\} \tag{3}$$

where $\mathbf{Co}\{\tilde{\mathcal{B}}_\delta\}$ denotes the convex hull of the n elements of $\tilde{\mathcal{B}}_\delta$.
Let us suppose that the subsets X_i may be represented as follows.

$$X_i = \left\{ x : \left\{ \begin{array}{l} x'\phi_{ik}(x)x \geq 0 \ , \quad \psi_{ik}(x) \geq 0 \ , \quad k = 1, \ldots, m_i \\ \Gamma_i(x)'x = 0 \end{array} \right. \right\} \ , \text{ for } \quad i = \{1, \ldots, n\} \tag{4}$$

where $\phi_{ik}(x) \in \mathbb{R}^{n_x \times n_x}$, $\Gamma_i(x) \in \mathbb{R}^{n_x}$, $\psi_{ik}(x) \in \mathbb{R}$ are given affine functions of x.
In order to introduce some auxiliary notation, we will represent the system (2) in a different form. Since $\delta_n = 1 - \sum_{i=1}^{n-1} \delta_i$, we get from (2):

$$\dot{x} = A_n x + \sum_{i=1}^{n-1} (A_i - A_n)\delta_i x + b_n + \sum_{i=1}^{n-1} (b_i - b_n)\delta_i \tag{5}$$

Finally, we introduce the auxiliary variables

$$\begin{array}{r} \pi_i = \delta_i x \ , \text{ for } \quad i = 1, \ldots, n-1 \\ \pi_n = 1 \\ \pi_{n+i} = \delta_i \ , \text{ for } \quad i = 1, \ldots, n-1 \end{array} \qquad \pi = \left[\begin{array}{c} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{2n-1} \end{array} \right] \in \mathbb{R}^{n_\pi}, \text{ where } \ n_\pi = (n-1)n_x + n \tag{6}$$

Therefore, we can express the system (5) as follows

$$\dot{x} \;=\; A_n x + \mathbf{A}\pi \ , \qquad (x,\pi) \;\in\; \mathcal{D} \tag{7}$$

with

$$\mathbf{A} \;=\; \left[\; (A_1 - A_n) \quad \cdots \quad (A_{n-1} - A_n) \quad b_n \quad (b_1 - b_n) \quad \cdots \quad (b_{n-1} - b_n) \;\right].$$

The notation $(x,\pi) \in \mathcal{D}$ is used to represent the relations between x, π, δ indicated in (6). Hereafter, the relation in (6) will be represented in a more compact form as indicated below.

$$\mathcal{D} \;=\; \left\{ (x,\pi) \;:\; \left[\; \Omega_1 \quad \Omega_2 \;\right] \begin{bmatrix} x \\ \pi \end{bmatrix} \;=\; 0 \right\} \tag{8}$$

where $\Omega_1 \in \mathbb{R}^{n_\Omega \times n_x}$, $\Omega_2 \in \mathbb{R}^{n_\Omega \times n_\pi}$ are affine matrix functions of $(x, \delta_1, \ldots, \delta_{n-1})$.
From the definition of β_δ in (3), it follows that

$$\delta \;\in\; \tilde{\mathcal{B}}_\delta \quad \text{if,} \quad \delta \;\in\; \mathcal{B}_\delta \ , \quad \text{and} \quad \delta\delta' = diag(\delta_i) \tag{9}$$

The condition $\delta\delta' - diag(\delta_i) \;=\; 0$ may be represented element by element as follows.

$$\begin{aligned} \delta_i\,(\delta_i - 1) \;\;&=\;\; 0 \ , \quad \forall\, i \;\in\; \{1, \ldots, n\} \\ \delta_i\,\delta_j \;\;&=\;\; 0 \ , \quad \forall\, i,j \;\in\; \{1, \ldots, n\} \end{aligned} \tag{10}$$

From (6) and (10), we get the following identities

$$\begin{array}{rclcl} (\delta_i - 1)\delta_i x & = & (\delta_i - 1)\pi_i & = & 0 \ , \quad \forall\, i \in \{1, \ldots, n-1\} \\ (\delta_i - 1)\delta_i & = & (\delta_i - 1)\pi_{n+i} & = & 0 \ , \quad \forall\, i \in \{1, \ldots, n-1\} \\ \delta_i\delta_j x & = & \delta_i\pi_j & = & 0 \ , \quad \forall\, i \neq j \ , \quad i,j \in \{1, \ldots, n-1\} \\ \delta_i\delta_j & = & \delta_i\pi_{j+n} & = & 0 \ , \quad \forall\, i \neq j \ , \quad i,j \in \{1, \ldots, n-1\} \end{array} \tag{11}$$

The above identities may be incorporated to (8) by rewriting them as new lines to be added to the matrices Ω_1 and Ω_2.
Since $\pi_n = 1$ and $\pi_{n+i} = \delta_i$, the following identities must hold too.

$$\begin{array}{lcl} x - x\pi_n & = & 0 \\ \pi_n - \sum_{i=1}^{n-1} \pi_{n+i} & = & 0 \\ \delta_i x - x\pi_{n+i} & = & 0 \ , \quad \forall\, i \in \{1, \ldots, n-1\} \\ \pi_i - x\pi_{n+i} & = & 0 \ , \quad \forall\, i \in \{1, \ldots, n-1\} \end{array} \tag{12}$$

Again, they may be incorporated to (8) as new lines to be added to Ω_1 and Ω_2.
Similarly, the condition

$$\Gamma_i(x)'\, x \;=\; 0 \tag{13}$$

may be also incorporated to (8) as a new line to be added to Ω_1 and Ω_2.

3 Stability Analysis

In this section, we state our main result in order to analyse the local stability of the system (1) in terms of linear matrix inequalities.

Theorem 1 *(Stability for Switched Systems)*
Consider the system (1), the associated system (2) and the assumptions **A 1-6**.
Let Θ_0 *be a given matrix affine function of* x *with the following structure*

$$\Theta_0 \,=\, \sum_{i=1}^{n_x} T_i x_i \;\in\; \mathbb{R}^{n_1 \times n_x} \tag{14}$$

where $T_i \in \mathbb{R}^{n_1 \times n_x}$, $i = 1, \ldots, n_x$, *are constant matrices with the same dimensions of* Θ_0 *and* x_i *are the entries of the vector* x.
From (14), consider the notation

$$\Theta_x \,=\, \sum_{i=1}^{n_x} T_i x r_i \tag{15}$$

where r_i *is the i-th row of* I_{n_x}.
Associated with the subsets X_i *in (4), let us introduce scalars* σ_{ik}, μ_{ik}, *for* $i = 1, \ldots, n$ *and* $k = 1, \ldots, m_i$, *to be determined and the following notation.*

$$\begin{aligned}
\Phi_i \,&=\, \sum_{k=1}^{m_i} \sigma_{ik}\phi_{ik} \in \mathbb{R}^{n_x \times n_x} \;;\quad \Psi_i \,=\, \sum_{k=1}^{m_i} \mu_{ik}\psi_{ik} \in \mathbb{R} \;,\; for \quad i = 1, \ldots, n \;; \\
\tilde{\Phi} \,&=\, diag\{\Phi_1, \ldots, \Phi_{n-1}\} \;; \qquad \tilde{\Psi} \,=\, diag\{\Psi_1, \ldots, \Psi_{n-1}\} \;; \\
E \,&=\, \begin{bmatrix} I_{n_x} & 0_{n_x \times n_1} & (-I_{n_x}) & \cdots & (-I_{n_x}) & 0_{n_x \times 1} & 0_{n_x \times 1} & \cdots & 0_{n_x \times 1} \\ 0_{1 \times n_x} & 0_{1 \times n_1} & 0_{1 \times n_x} & \cdots & 0_{1 \times n_x} & 1 & (-1) & \cdots & (-1) \end{bmatrix} \;; \\
\Phi \,&=\, diag\{\; 0_{n_x}, \;\; 0_{n_1}, \;\; \tilde{\Phi}, \;\; 0, \;\; \tilde{\Psi} \;\} + E' \begin{bmatrix} \Phi_n & 0_{n_x \times 1} \\ 0_{1 \times n_x} & \Psi_n \end{bmatrix} E \;.
\end{aligned} \tag{16}$$

Associated with the system representation (7), (8) and the matrices Θ_0, Θ_x *from (14), (15), consider the additional notation.*

$$\begin{aligned}
A_{a_1} \,&=\, \begin{bmatrix} I_{n_x} \\ \Theta_0 + \Theta_x \end{bmatrix} \begin{bmatrix} A_n & 0_{n_x \times n_1} \end{bmatrix} \;;\quad A_{a_2} \,=\, \begin{bmatrix} I_{n_x} \\ \Theta_0 + \Theta_x \end{bmatrix} \mathbf{A} \;; \\
C_x \,&=\, \begin{bmatrix} x_2 & -x_1 & 0 & \cdots & 0 \\ 0 & x_3 & -x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n_x} & -x_{n_x-1} \end{bmatrix} \;;\quad C \,=\, \begin{bmatrix} C_x & 0 \\ \Theta_0 & -I \end{bmatrix} \;;\quad \tilde{\Omega} \,=\, \begin{bmatrix} \Omega_1 & 0_{n_\Omega \times n_1} & \Omega_2 \\ & C \;\; 0 & \end{bmatrix}
\end{aligned} \tag{17}$$

where the matrix C_x *is an auxiliary matrix borrowed from [21], lemma 2.1.*
Then the system (2) is locally exponentially stable if there exist matrices P, L, M, *with the same dimensions of* A_{a_1}, C' *and* $\tilde{\Omega}'$ *respectively, and positive scalars* σ_{ik}, μ_{ik}, *such that the following LMIs are satisfied at the vertices of the meta-polytope* $\mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta$*:*

$$\begin{aligned}
&P + LC + C'L' \,>\, 0 \;, \quad P \,=\, P' \\
&\begin{bmatrix} (A_{a_1}'P + PA_{a_1}) & PA_{a_2} \\ A_{a_2}'P & 0 \end{bmatrix} + M\tilde{\Omega} + \tilde{\Omega}'M' + \Phi \,<\, 0
\end{aligned} \tag{18}$$

In affirmative case, the function $v(x) = x'\mathcal{P}(x)x$, *with* $\mathcal{P}(x)$ *given below, is a local Lyapunov function for the origin of the system (2).*

$$\mathcal{P}(x) \,=\, \begin{bmatrix} I_{n_x} \\ \Theta_0 \end{bmatrix}' P \begin{bmatrix} I_{n_x} \\ \Theta_0 \end{bmatrix} \tag{19}$$

□□

Proof

Consider the system (2) and the vector π as in (6). Define the following auxiliary vectors.

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} , \quad \xi_1 = \begin{bmatrix} x \\ \Theta_0 \, x \end{bmatrix} , \quad \xi_2 = \pi . \tag{20}$$

Let $D_1 \in \mathbb{R}^{n_x \times (n_x+n_1)}$ be a matrix such that $D_1\xi_1 = x$ and define $D_2 = \begin{bmatrix} D_1 & 0_{n_x\times(n_\pi-n_x-n_1)} \end{bmatrix}$. Suppose that (18) is satisfied at the vertices of the polytope $\mathcal{B}$. Then, by convexity it is also satisfied $\forall\, x \in \mathcal{B}_x$, $\delta \in \mathcal{B}_\delta$ and then $\forall\, \delta \in \tilde{\mathcal{B}}_\delta$.

Since the inequalities in (18) are strict, then for some sufficient small positive scalars ϵ_1, ϵ_2, it is possible to add the term $-\epsilon_1 D_1' D_1$ to the first LMI and $\epsilon_2 D_2' D_2$ to the second LMI without changing the sign of the inequalities, i.e., the following conditions are still satisfied.

$$\begin{array}{l} P + LC + C'L' - \epsilon_1 D_1' D_1 \geq 0 \ , \\ \begin{bmatrix} (A_{a_1}' P + P A_{a_1}) & P A_{a_2} \\ A_{a_2}' P & 0 \end{bmatrix} + M\tilde{\Omega} + \tilde{\Omega}' M' + \Phi + \epsilon_2 D_2' D_2 \leq 0 \end{array} \tag{21}$$

Pre and post multiply the first LMI in (21) by ξ_1' and its transpose respectively, and the second LMI by ξ' and its transpose respectively. These two operations lead the following to be satisfied.

$$\begin{array}{l} \xi_1' P \xi_1 \geq \epsilon_1 x'x \ , \quad \forall\, x \in \mathcal{B}_x \ : C\xi_1 = 0 \\ \xi_1'(A_{a_1}' P + P A_{a_1})\xi_1 + \xi_1' P A_{a_2} \xi_2 + \xi_2' A_{a_2}' P \xi_1 + \xi' \Phi \xi \leq -\epsilon_2 x'x \ , \forall x \in \mathcal{B}_x, \delta \in \tilde{\mathcal{B}}_\delta \ : \tilde{\Omega}\xi = 0 \end{array} \tag{22}$$

Notice that, by construction, we have $C_x\, x = 0$. Moreover, the constraint $C\,\xi_1 = 0$ implies that ξ_1 has the form $\xi_1' = \begin{bmatrix} x' & x'\Theta_0' \end{bmatrix}$. Similarly the constraint $\tilde{\Omega}\xi = 0$ is equivalent to

$$C\,\xi_1 = 0 \quad \text{and} \quad \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0$$

Observe that (8) represents the identities (6), (11), (12) and (13).

As $\xi_1' = \begin{bmatrix} x' & x'\Theta_0' \end{bmatrix}$, the first expression of (22) leads to

$$v(x) = \xi_1' P \xi_1 = x' \begin{bmatrix} I_{n_x} \\ \Theta_0 \end{bmatrix}' P \begin{bmatrix} I_{n_x} \\ \Theta_0 \end{bmatrix} x = x' \mathcal{P}(x)\, x \geq \epsilon_1 x'x \ , \quad \forall\, x \in \mathcal{B}_x \ : \ C\xi_1 = 0 \tag{23}$$

where $\mathcal{P}(x)$ is the state dependent matrix given in (19).

Since x belongs to the polytope $\mathcal{B}_x$, the entries of the matrix C in the first LMI of (18) are bounded. Then there exists a sufficiently large positive scalar ϵ_3 such that $\epsilon_3 I_{n_x} \geq (P + LC + C'L')$. Thus, $\epsilon_3 \xi_1' \xi_1 = \epsilon_3(x'x + x'\Theta_0'\Theta_0 x) \geq \xi_1 P \xi_1 = x'\mathcal{P}(x)x$. Keeping in mind that x belongs to a polytope there exists another sufficient large positive scalar ϵ_4 such that $\epsilon_4 I_{n_x} \geq \Theta_0'\Theta_0$. Hence $x'\mathcal{P}(x)x \leq \epsilon_3(1+\epsilon_4)x'x$, which together with (23) yields

$$\epsilon_1 x'x \leq v(x) = x'\mathcal{P}x \leq \epsilon_3(1+\epsilon_4)x'x \ , \quad \forall\, x \in \mathcal{B}_x \tag{24}$$

From (16) and the identity $\delta_n = 1 - \sum_{i=1}^{n-1} \delta_i$, note that

$$E\xi = \begin{bmatrix} x - \sum_{i=1}^{n-1} \pi_i \\ \\ \pi_n - \sum_{i=1}^{n-1} \pi_{n+i} \end{bmatrix} = \begin{bmatrix} \delta_n x \\ \\ \delta_n \end{bmatrix}$$

and hence $\xi'\Phi\xi$ in the last inequality of (22) yields

$$\begin{aligned}
\xi'\Phi\xi &= \sum_{i=1}^{n-1} \pi_i^{'}\Phi_i\pi_i + \left(x - \sum_{i=1}^{n-1}\pi_i\right)^{'} \Phi_n \left(x - \sum_{i=1}^{n-1}\pi_i\right) + \\
&+ \sum_{i=1}^{n-1} \pi_{n+i}^{'}\Psi_i\pi_{n+i} + \left(\pi_n - \sum_{i=1}^{n-1}\pi_{n+i}\right)^{'} \Psi_n \left(\pi_n - \sum_{i=1}^{n-1}\pi_{n+i}\right) \\
&= \sum_{i=1}^{n-1} \delta_i^2 x'\Phi_i x + \delta_n^2 x'\Phi_n x + \sum_{i=1}^{n-1}\delta_i^2\Psi_i + \delta_n^2\Psi_n \geq 0 \ , \quad \forall \ x \in \mathcal{B}_x \ , \quad \delta \in \tilde{\mathcal{B}}_\delta
\end{aligned}$$

Since $\xi'\Phi\xi \geq 0$, the second inequality of (22) implies the following

$$\xi_1^{'}(A_{a_1}^{'}P + PA_{a_1})\xi_1 + \xi_1^{'}PA_{a_2}\xi_2 + \xi_2^{'}A_{a_2}^{'}P\xi_1 \leq -\epsilon_2 x'x \ , \ \ \forall \ x \in \mathcal{B}_x \ , \ \ \delta \in \tilde{\mathcal{B}}_\delta \tag{25}$$

Using the fact that $\dot{\Theta}_0\, x \ = \ \Theta_x\dot{x}$ and the identities

$$\begin{aligned}
\dot{x} &= A_n x + \sum_{i=1}^{n-1}\delta_i(A_i - A_n)x + b_n + \sum_{i=1}^{n-1}\delta_i(b_i - b_n) \ , \\
\xi_1^{'}PA_{a_1}\xi_1 &= x'\mathcal{P}(x)\left[\begin{array}{cc} A_n & 0 \end{array}\right]\xi_1 + x'\left[\begin{array}{c} I_{n_x} \\ \Theta_0 \end{array}\right]^{'} P \left[\begin{array}{c} I_{n_x} \\ \dot{\Theta}_0 \end{array}\right] x \ , \quad \text{and} \\
\xi_1^{'}PA_{a_2}\xi_2 &= x'\mathcal{P}(x)\mathbf{A}\xi_2
\end{aligned}$$

we can show that

$$\begin{aligned}
\xi_1^{'}(A_{a_1}^{'}P + PA_{a_1})\xi_1 + \xi_1^{'}PA_{a_2}\xi_2 + \xi_2^{'}A_{a_2}^{'}P\xi_1 &= x'\dot{\mathcal{P}}(x)x + 2x'\mathcal{P}(x)\left(A_n x\sum_{i=1}^{n-1}(A_i - A_n)\pi_{i+1}\right) + \\
&+ 2x'\mathcal{P}(x)\left(b_n + \sum_{i=1}^{n-1}(b_i - b_n)\pi_{n+i+1}\right) \leq -\epsilon_2 x'x
\end{aligned}$$

Keeping in mind that $\delta_n \ = \ 1 - \sum_{i=1}^{n-1}\delta_i$, and considering the definition (6) of the vector π, we have

$$A_n x + \sum_{i=1}^{n-1}(A_i - A_n)\pi_i = \left(\sum_{i=1}^{n} A_i\delta_i\right)x \quad \text{and} \quad b_n + \sum_{i=1}^{n-1}(b_i - b_n)\pi_{i+n} \ = \ \sum_{i=1}^{n}\delta_i b_i$$

Then, we can conclude that

$$\begin{aligned}
\dot{v}(x) &= x'\dot{\mathcal{P}}(x)x + x'\left(\sum_{i=1}^{n}\delta_i A_i\right)^{'}\mathcal{P}(x)x + x'\mathcal{P}(x)\left(\sum_{i=1}^{n}\delta_i A_i\right)x + \\
&+ 2\left(\sum_{i=1}^{n}\delta_i b_i\right)^{'}\mathcal{P}(x)x \leq -\epsilon_2 x'x \ , \ \ \forall \ x \in \mathcal{B}_x \ , \ \ \delta \in \tilde{\mathcal{B}}_\delta
\end{aligned} \tag{26}$$

Finally, the relations (24) and (26) show that the origin of the system (2) is locally exponentially stable and $v(x) \ = \ x'\,\mathcal{P}(x)\,x$ is a Lyapunov function for this system.

□

It is important to observe that the dimensions of π in (6) and Θ_0 in (14) play an important role in the conservativeness of the method because they affect the dimensions of the matrices P, L, M which are the decision variables to be tuned. More variables to tune is likely to reduce the conservatism of the method but it increases the computational effort.

Remark 1 *(Sliding Modes)*
Eventually the state trajectory reaches the boundary of ρ *adjacent subsets* X_i *and does not leave it, characterizing the concept of sliding modes. However this behaviour is not allowed in system (1) with Assumption* **A6**.
On the other hand, it is possible to deal with sliding modes using the Filippov's Approach [20]. In this way, an interesting result was proposed in [16], Propositon 1, that is summarized as follows:

> If the Lyapunov function is non-increasing along the system trajectories on a commom boundary $\mathcal{X} \triangleq \{ X_i \cap \ \ldots \ \cap X_{i+\rho} \}$, for ρ adjacent subsets, then the resulting sliding mode is stable, i.e.
>
> $$\dot{v}(x) \leq 0 , \quad \forall\, x \in \mathcal{X} \tag{27}$$
>
> Moreover, if
>
> $$\dot{v}(x) < 0 , \quad \forall\, x \in \mathcal{X} \tag{28}$$
>
> the asymptotic stability is guaranteed.

By requiring that the Lyapunov function also satisfies the condition (27), or (28), the stability analysis can be extended to handle systems with sliding modes. Observe that this requirement can be recast as an LMI problem. In fact, more work in this topic is carrying out by the authors.

□

4 Domain of Attraction

The theorem (1) establishes sufficient conditions for the local stability of the origin of the system (1). In this section we address the problem of estimating a suitable domain of attraction of the origin of the system. The basic idea is to compute a Lyapunov function such that its largest level set inside a given polytope $\mathcal{B}_x$ is maximized in a sense to be specified later on.
For this purpose let us consider that the polytope $\mathcal{B}_x$ will be described by linear inequalities as follows

$$\mathcal{B}_x = \left\{ x : a_k^{'} x \leq 1 , \ k = 1, \ldots , n_e \right\}$$

where the vectors a_k define the n_e edges of the polytope. Remember that $\mathcal{B}_x$ may be equivalently represented by its vertices.
Consider the region

$$\Upsilon = \left\{ x : v(x) = x' \mathcal{P}(x) x \leq 1 \right\} \tag{29}$$

as an estimate of the domain of attraction. Note that Υ is not an usual ellipsoid because the Lyapunov function, $v(x)$, is not a quadratic function of x.

We search conditions to insert the region Υ in the polytope $\mathcal{B}_x$, i.e., we would like to find conditions to render Υ a positively invariant region.
For this purpose, consider the auxiliary vectors

$$c_k = \begin{bmatrix} a_k' & 0 \end{bmatrix}' \in \mathbb{R}^{n_x+n_1} \ , \quad \text{for } k = 1, \ldots, n_e \tag{30}$$

and the vector ξ_1 defined in (20).
The region $\Upsilon \subset \mathcal{B}_x$ if $\forall\, x \in \Upsilon$ then $x \in \mathcal{B}_x$. This inclusion may be formulated as

$$c_k' \, \xi_1 \leq 1 \ , \quad \forall\ \xi_1 : \xi_1' P \xi_1 \leq 1$$

From [17] we may recast the above condition as the LMI feasibility problem below.

$$\begin{bmatrix} 1 & c_k' \\ c_k & (P + LC + C'L') \end{bmatrix} \geq 0 \ , \quad \text{for } k = 1, \ldots, n_e \tag{31}$$

To sum up, Υ is contained in the polytope $\mathcal{B}_x$ if there exists a solution to (18) with the additional constraints (31). The estimate of the domain of attraction may be maximized in several ways [17], [21, 22].
In this work, we will consider the following optimization problem.

$$min \ \lambda : \left\{ \begin{array}{l} (18),\ (31) \text{ and} \\ \lambda - \ \text{trace}\ (P + LC + C'L') \geq 0 \end{array} \right. \tag{32}$$

5 Numerical Examples

In this section, we will analyse two numerical examples to illustrate the potential of the proposed method. In the first one, borrowed from [23], we are interested in analysing the local stability of a piecewise linear system. In the other one, borrowed from [24], we address the problem of estimating a domain of attraction of a feedback linear system with saturation.

Example 5.1 *(Stability Analysis)*
Consider the following system

$$\dot{x} = \left\{ \begin{array}{ll} A_1 x, & se\ x_1 \leq 0 \\ A_2 x, & se\ x_1 \geq 0 \end{array} \right. \tag{33}$$

with

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}.$$

Rewriting this system in terms of the representation (2), we get

$$\dot{x} \ = \ \delta_1 A_1 x \ + \ \delta_2 A_2 x, \tag{34}$$

where δ_1 and $\delta_2 = 1 - \delta_1$ are the logical variables. In order to apply the theorem 1 we need to choose the matrix Θ_0, which is associated with the structure of the Lyapunov function. See [21] for more details.

$$\Theta_0 \ = \ \begin{bmatrix} x_1 I_2 \\ x_2 I_2 \end{bmatrix} \ , \quad \textit{and consequently} \quad n_1 \ = \ 4$$

So we have

$$\Theta_0 x = \left[\begin{array}{cccc} x_1^2 & x_1 x_2 & x_1 x_2 & x_2^2 \end{array} \right]' \quad \textit{and} \quad \pi_1 = \delta_1 x \tag{35}$$

In this example, note that the system is piecewise linear, i.e., $b_i = 0, \forall\, i$. *In this case the expressions of the theorem 1 may be simplified, as it can be seen from (5), (6), (7). It turns out that it is no longer necessary to use the auxiliary vectors* $\pi_n, \cdots, \pi_{2n-1}$ *and therefore the column dimensions of the auxiliary matrices can be reduced. In particular the matrices* Ω_1, Ω_2 *are given by*

$$\Omega_1 = \left[\begin{array}{c} 0_2 \\ \delta_1 I_2 \end{array} \right] \quad \textit{and} \quad \Omega_2 = \left[\begin{array}{c} (\delta_1 - 1) I_2 \\ -I_2 \end{array} \right]$$

For this example the regions X_i *in (4) are characterized as follows*

$$x'\phi_{11}(x)x \geq 0 \Rightarrow \phi_{11}(x) = \left[\begin{array}{cc} -x_1 & 0 \\ 0 & -x_1 \end{array} \right] \textit{ and } x'\phi_{21}(x)x \geq 0 \Rightarrow \phi_{21}(x) = \left[\begin{array}{cc} x_1 & 0 \\ 0 & x_1 \end{array} \right].$$

In this case $\Gamma_i = 0 \;\; \forall\, i$ *and* $\psi_{ik}(x) = 0 \;\; \forall\, i,k$. *Therefore the construction of* Φ *in the theorem 1 leads to*

$$\Phi_1 = \sigma_{11}\phi_{11}(x), \quad \Phi_2 = \sigma_{21}\phi_{21}(x), \;\; \textit{and} \;\; \tilde{\Phi} = \Phi_1, \quad \tilde{\Psi} = 0, \quad \Psi_2 = 0$$

and then we have:

$$\Phi = diag\{0_2,\ 0_4,\ \tilde{\Phi}\} + E' \left[\begin{array}{cc} \Phi_2 & 0_{2\times 1} \\ 0_{1\times 2} & 0 \end{array} \right] E$$

with

$$diag\{0_2,\ 0_4,\ \tilde{\Phi}\} = \sigma_{11} \left[\begin{array}{cc} 0_6 & 0_{6\times 2} \\ 0_{2\times 6} & -x_1 I_2 \end{array} \right],$$

and

$$E' \left[\begin{array}{cc} \Phi_2 & 0_{2\times 1} \\ 0_{1\times 2} & 0 \end{array} \right] E = \sigma_{21} \left[\begin{array}{ccc} x_1 I_2 & 0_{2\times 4} & -x_1 I_2 \\ 0_{4\times 2} & 0_4 & 0_{4\times 2} \\ -x_1 I_2 & 0_{2\times 4} & x_1 I_2 \end{array} \right].$$

Observe that

$$A_{a_1} = \left[\begin{array}{cc} A_2 & 0 \\ (\Theta_0 + \Theta_x) A_2 & 0 \end{array} \right], \quad A_{a_2} = \left[\begin{array}{c} A_1 - A_2 \\ (\Theta_0 + \Theta_x)(A_1 - A_2) \end{array} \right]$$

where

$$\Theta_0 + \Theta_x = \left[\begin{array}{cc} x_1 & 0 \\ 0 & x_1 \\ x_2 & 0 \\ 0 & x_2 \end{array} \right] + \left[\begin{array}{cc} x_1 & 0 \\ x_2 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{array} \right].$$

We consider the polytope

$$\mathcal{B}_x = \left\{ x = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] : |x_i| \leq \alpha \right\}$$

where $\alpha > 0$ *is a scalar to be determined such that* $\mathcal{B}_x$ *will be as large as possible. The vertices of the polytope* $\mathcal{B}_x$ *are*

$$V_i = \left[\begin{array}{c} \pm\alpha \\ \pm\alpha \end{array} \right], \quad i = 1, \cdots, 4.$$

The polytope $\mathcal{B}_\delta$ *is given by:*

$$\mathcal{B}_\delta = \mathbf{Co} \left\{ \left[\begin{array}{c} 0 \\ 1 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \right\}.$$

Then the polytope $\mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta$ has $4 \times 2 = 8$ vertices.
Using the Theorem 1 we conclude that the system (33) is locally exponentially stable with $\alpha =$ 100.
The Lyapunov function for this α is given by $v(x) = x'\mathcal{P}(x)x$ with $\mathcal{P}(x)$ as in (19) and matrix P given by

$$P = \begin{bmatrix} 7279067.9 & 434435.48 & 22280.18 & 657.259 & 7500.484 & -554.009 \\ 434435.48 & 1506838.5 & 6199.410 & 5120.643 & 1763.322 & 3005.408 \\ 22280.18 & 6199.410 & 781.2508 & 414.7109 & -562.9214 & 24.38704 \\ 657.2597 & 5120.643 & 414.7109 & 1. & -98.67125 & 25.63713 \\ 7500.484 & 1763.322 & -562.9214 & -98.67125 & 367.0776 & -56.06310 \\ -554.009 & 3005.408 & 24.38704 & 25.63713 & -56.06310 & 30.42994 \end{bmatrix}.$$

The large values obtained for α suggests that the method could be used to show the semi-global stability [8] of the system. For this particular example it is known that the system is glogbally stable [23].

$\triangle\triangle\triangle$

Example 5.2 *(Domain of attraction of a saturated linear system) Consider the following saturated linear system.*

$$\begin{cases} \dot{x} = Ax + b\,sat(u) \\ u = K'x \end{cases} \quad for\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \ and\ K = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{36}$$

where $sat(\cdot)$ denotes the normalized unit saturation function given by

$$sat(u) \triangleq \begin{cases} 1 & & u > 1 \\ u & for & |u| \leq 1 \\ -1 & & u < -1 \end{cases}$$

The problem of concern in this example is to compute an estimate of the domain of attraction. For this purpose, we consider the following.
The saturated system is written as a switched system, as follows.

$$\dot{x} = \begin{cases} Ax - b & x \in X_1 \\ (A + bK')\,x & x \in X_2 \\ Ax + b & x \in X_3 \end{cases}$$

Note that the state space is partitioned in three polyhedral cells corresponding to negative saturation (X_1), linear operation (X_2), and positive saturation (X_3) respectively.
As in [24], the states of the system are constrained to lie in the following set $\mathbb{X} = \{x : |K'x| \leq 5\}$.
The above system, in terms of the representation (2), is given by $\dot{x} = (A_1\delta_1 + A_2\delta_2 + A_3\delta_3)x + b_1\delta_1 + b_2\delta_2 + b_3\delta_3$, where the matrices A_1, A_2, A_3 and vectors b_1, b_2, b_3 are given by

$$A_1 = A_3 = A\ , \quad A_2 = \begin{bmatrix} 0 & 1 \\ -9 & -5 \end{bmatrix}, \quad b_1 = -b\ , \quad b_3 = b\ , \ and \quad b_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix Θ_0 is the same of the example (5.1).

The polytope $\mathcal{B}_x$ is defined by the following vertices.

$$\left\{ \begin{bmatrix} \beta \\ -4\beta \end{bmatrix}, \begin{bmatrix} 3\beta \\ -2\beta \end{bmatrix}, \begin{bmatrix} -\beta \\ 4\beta \end{bmatrix}, \begin{bmatrix} -3\beta \\ 2\beta \end{bmatrix} \right\}$$

where $\beta > 0$ is a given scalar.
The polytope $\mathcal{B}_\delta$ is the same of the example (5.1). The regions X_i are defined by the following functions.

- $X_1 \rightarrow \psi_{11}(x) = -(1 + K^{'}x)$, $\psi_{12}(x) = K^{'}x + 5$, $\phi_{11}(x) = I_2\psi_{11}(x)$ *and* $\phi_{12}(x) = I_2\psi_{12}(x)$.
- $X_2 \rightarrow \psi_{21}(x) = (1 + K^{'}x)$, $\psi_{22}(x) = 1 - K^{'}x$, $\phi_{21}(x) = I_2\psi_{21}(x)$ *and* $\phi_{22}(x) = I_2\psi_{22}(x)$.
- $X_3 \rightarrow \psi_{31}(x) = (K^{'}x - 1)$, $\psi_{32}(x) = 5 - K^{'}x$, $\phi_{31}(x) = I_2\psi_{31}(x)$ *and* $\phi_{32}(x) = I_2\psi_{32}(x)$.

In the above three cells we consider $\Gamma_i(x) = 0_{2\times 1}$. The algebraic constraints (11) and (12) are used to construct the matrices Ω_1, Ω_2 similarly to the example 5.1.
The estimate of the domain of attraction is the region $\Upsilon = \{x : x'\mathcal{P}(x)x \leq 1\}$, where the state dependent matrix $\mathcal{P}(x)$ has the structure indicated in (19).
Solving the optimization problem (32) for $\beta = 2.2$, we obtain the following matrix

$$P = \begin{bmatrix} 0.0797691 & 0.0420854 & -0.0000013 & -79275742. & 78938897. & 1.028E+08 \\ 0.0420854 & 0.0438128 & 336845. & -1.028E+08 & 0. & 3.527E-07 \\ -0.0000013 & 336845. & 0.0002496 & 8167605.7 & -8167605.7 & 7.336E-07 \\ -79275742. & -1.028E+08 & 8167605.7 & 516887.76 & -258444.38 & 74520.442 \\ 78938897. & 0. & -8167605.7 & -258444.38 & 1. & -74520.442 \\ 1.028E+08 & 3.527E-07 & 7.336E-07 & 74520.442 & -74520.442 & 0.0002503 \end{bmatrix}$$

The figure 1 shows an estimate of the domain of attraction using the proposed method. For comparative purposes it is indicated the regions estimated with the method in [24], via circle and Popov criteria, and another one using the Piecewise Quadratic (PwQ) approach from [1]. In this example we note that our method finds a lightly better region than the circle and Popov criteria and surprisingly the piecewise quadratic approach from [1] gives the worst estimate.

$\triangle\triangle\triangle$

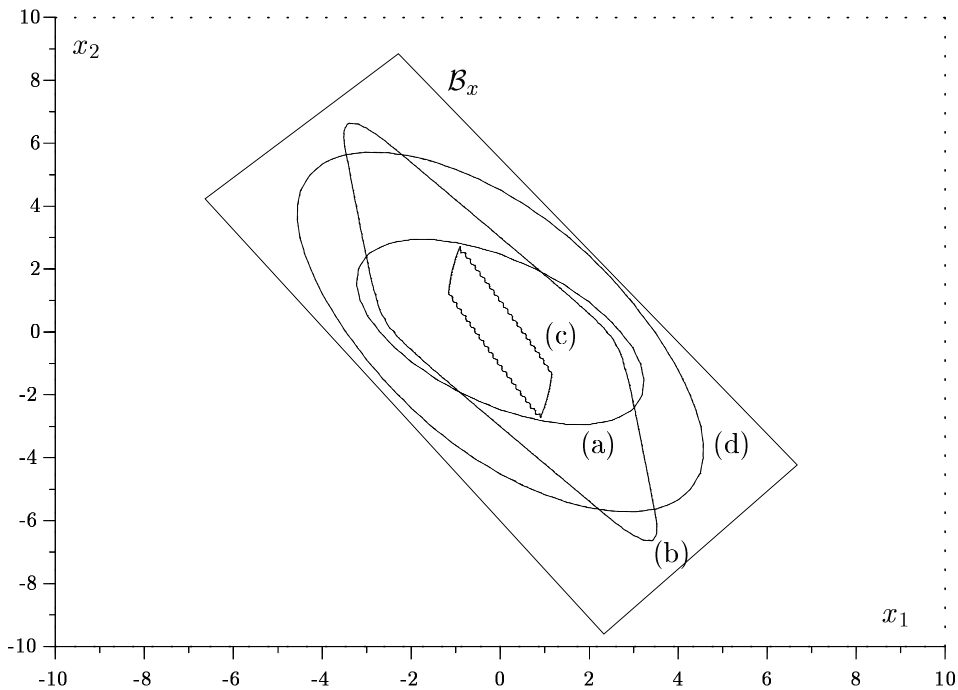

Figure 1: Estimates of the domain of attraction of the system (36). (a) Circle criterion, (b) Popov criterion, (c) PwQ Approach and (d) Proposed Method.

6 Conclusions

In this work it was studied the stability of switched systems of the type $\dot{x} = A(\delta)x + b(\delta)$, where x is the continuous state and δ is a vector of logical variables with a switching logic that depends on the continuous state only. The approach used here to analyse the stability and domain of attraction is based on a Lyapunov function, $v(x) = x'\mathcal{P}(x)x$, where the matrix $\mathcal{P}(x)$ is dependent on the continuous state. The stability conditions established are expressed in terms of LMI's and the results are illustred throught numerical examples that confirm the potential of the proposed method.
Different from other methods [19, 1] that use a piecewise Lyapunov function, in this paper it is used a single, but more complex, Lyapunov function for all state space partitions.
In our opinion, one of the advantages of this approach is that the use a single Lyapunov function is likely an easy way to estimate the domain of attraction of the switched/hybrid system. As future work, besides the extension to the discrete time case and to performance analysis problems, we intend to apply these results for more complex systems, for instance that whose switching rules depend on continuous and logical variables and that the local continuous dynamics have nonlinear behaviour. The best way to maximize the estimate of the domain of attraction is another factor to be explored.

Acknowledgments

The authors would like to thank Romeu Reginatto for his valuable comments on this paper. The numerical results in this paper were obtained with the LMI toolbox of the free software Scilab, [25].

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